SEQUENCES OF POWERS IN GENETIC ALGEBRAS

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1. *Introduction*

Certain non-associative algebras have important applications in theoretical Mendelian genetics. The following definitions and summary of the theory are taken from papers by Etherington [2]–[7], who initiated their study. A *baric algebra* is one which admits a non-trivial homomorphism, $x \to \omega(x)$, into its coefficient field. $\omega(x)$ is called the *weight* of x. In this paper we shall only be concerned with commutative algebras over the real numbers. Clearly, the basis of a baric algebra may be so chosen that one of its elements, say c_0, has weight unity, and the remainder, $c_1, \ldots, c_n$ have weight zero. The set of elements of unit weight is closed with respect to multiplication, while the set of elements of zero weight is an ideal N, the nil ideal.

In a non-associative system there is no unambigious k-th power of an element. In genetic algebras the following sequences are important: the principal powers x^k, defined by

$$x^1 = x, \quad x^k = x^{k-1}\, x,$$

and the plenary powers $x^{[k]}$ defined by

$$x^{[1]} = x, \quad x^{[k]} = x^{[k-1]}\, x^{[k-1]}.$$

Let the rank equation connecting the principal powers of x be

$$x^k + a_1\, x^{k-1} + \ldots + a_{k-1}\, x = 0.$$

In general the a_i will depend on x, but if they depend on x only through $\omega(x)$, the baric algebra is called a *train algebra*, of principal rank k. In view of the baric property this condition is equivalent to requiring that for each element of unit weight, the rank equation should have constant coefficients, and the roots of the corresponding scalar equation

$$\lambda^{k-1} + a_1 \lambda^{k-2} + \ldots + a_{k-1} = 0$$

are called the principal train roots of the algebra.

A *special train algebra* is a baric algebra in which N is nilpotent, and all the principal powers of N are ideals. Such an algebra is necessarily a train algebra.

Suppose that in a train algebra a certain sequence of powers is of interest, perhaps because of some genetic application. The question arises as to whether this sequence also satisfies an equation of degree k', say,

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such that for elements of unit weight, the coefficients are constant. If so, the sequence is said to be a *train* of order k', the equation is called the *train equation* and the roots of the corresponding scalar equation the *train roots* of the sequence, for the algebra.

Etherington showed [4], that all train algebras of ranks 1, 2 and 3 are special train algebras, the plenary powers form trains, and determined their plenary train roots. Reiersøl [13], proved that this result holds for algebras corresponding to sets of n linked loci, and showed how the plenary train roots may be computed by recursion with respect to n. In a previous paper [11], I proved the same result for a class of algebras including those corresponding to polysomic inheritance, and showed how their plenary train roots could be computed. In this paper I show that the result holds for all commutative special train algebras containing idempotent elements, and show how the distinct values among their plenary train roots, and upper bounds for their multiplicities may be obtained by recursion on the dimension of the algebra.

2. *Definitions*

Let A_n denote the special train algebra over a space of dimension $n-1$ whose basis in the canonical form obtained in [5] is $c_0, c_1, \ldots, c_n$, in which a typical element of unit weight is

$$x = c_0 + u_1 c_1 + \ldots + u_n c_n. \tag{1}$$

Its multiplication table in the form given by Gonshor [9] is

$$c_i c_j = \sum_{k=0}^{n} \lambda_{ijk} c_k, \tag{2}$$

where

$$\left.\begin{array}{ll} \lambda_{000} = 1 & \\ \lambda_{0jk} = 0, & k < j \\ \lambda_{ijk} = 0, & i, j > 0 \text{ and } k \leqslant \max(i, j), \end{array}\right\} \tag{3a}$$

The λ_{0jj} include the principal train roots, possibly with repetitions, and may include $\frac{1}{2}$ even when it is not a principal train root. If A_n contains an idempotent, it can be taken as c_0, in which case

$$\lambda_{00k} = 0, \quad k > 0. \tag{3b}$$

Conditions for this to be true are given by Gonshor. Henceforth we will assume that the algebra does contain an idempotent. It will often be convenient, without introducing additional notation, to refer to x as a row vector of coefficients, so that the element (1) is denoted by $(1, u_1, \ldots, u_n)$. Let E_n be the operator on A_n which transforms x into x^2. It will be written on the right of the element operated on, to maintain consistency with possible matrix representations, and it will often be equally convenient to think of it as operating on the coefficients u_i.

3. *Algebras of low rank*

A_1 has multiplication table

	c_0	c_1
c_0	c_0	$\lambda_{011} c_1$
c_1		0 ,

and if $\lambda_{011} \neq \frac{1}{2}$, its plenary train roots are 1, $2\lambda_{011}$. If $\lambda_{011} = \frac{1}{2}$, unity is the unique plenary train root (see [4]). To illustrate the method used in this paper, let us determine the plenary train roots of A_2 which has multiplication table

	c_0	c_1	c_2
c_0	c_0	$\lambda_{011} c_1 + \lambda_{012} c_2$	$\lambda_{022} c_2$
c_1		$\lambda_{112} c_2$	0
c_2			0 .

The square of an element of unit weight is

$$xE_2 = x^2 = c_0 + 2\lambda_{011} u_1 c_1 + (2\lambda_{012} u_1 + \lambda_{112} u_1^2 + 2\lambda_{022} u_2) c_2,$$

or expressed as a transformation of the u_i;

$$\left.\begin{aligned} 1\, E_2 &= 1, \\ u_1 E_2 &= 2\lambda_{011} u_1, \\ u_2 E_2 &= 2\lambda_{012} u_1 + \lambda_{112} u_1^2 + 2\lambda_{022} u_2. \end{aligned}\right\} \tag{4}$$

Let us now associate with A_2 a space B_2 of vectors (v_0, v_1, v_2, v_3), and define a mapping R of the plane of unit weight in A_2 into a variety V_2 in B_2 by

$$\begin{aligned} (1, u_1, u_2) R &= (1, v_1, v_2, v_3) \\ &= (1, u_1, u_1^2, u_2). \end{aligned}$$

E_2 induces a transformation $\hat{E}_2$ of the v_i;

$$\begin{aligned} 1\, \hat{E}_2 &= 1, \\ v_1 \hat{E}_2 &= 2\lambda_{011} v_1, \\ v_2 \hat{E}_2 &= 4\lambda_{011}^2 v_2, \\ v_3 \hat{E}_2 &= 2\lambda_{012} v_1 + \lambda_{112} v_2 + 2\lambda_{022} v_3. \end{aligned}$$

This can be extended to a linear transformation of the whole of B_2 with matrix

$$\hat{E}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\lambda_{011} & 0 & 2\lambda_{012} \\ 0 & 0 & 4\lambda_{011}^2 & \lambda_{112} \\ 0 & 0 & 0 & 2\lambda_{022} \end{pmatrix}.$$

The operator E_2 acting on elements of unit weight in A_2 can now be seen to correspond to $\hat{E}_2$ acting on their images in B_2, in the sense that

$$(xE_2)\,R = (xR)\,\hat{E}_2. \tag{5}$$

Hence if $f(\hat{E}_2)$ is a polynomial operator which annihilates B_2, $f(E_2)$ will annihilate A_2. Hence the minimal polynomial of $\hat{E}_2$ contains as a factor a polynomial which corresponds to the plenary rank equation of A_2. In view of its upper triangular form, it can be seen that the proper values of $\hat{E}_2$, and hence the plenary train roots of A_2, are included in the set 1, $2\lambda_{011}$, $4\lambda_{011}^2$, $2\lambda_{022}$.

If this set contains superfluous elements, which it will for certain combinations of values of the λ_{ijk}, it is sometimes possible to modify B_2, V_2, R and $\hat{E}_2$ so that they are removed from the principal diagonal of $\hat{E}_2$. For instance:

(i) If $\lambda_{011} = \lambda_{022}$, $2\lambda_{011}$ is an unrepeated plenary train root. Hence the last coordinate of B_2, and the last row and column of $\hat{E}_2$ can be deleted. (5), and the result, still hold.

(ii) If $\lambda_{022} = \frac{1}{2}$, unity is an unrepeated plenary train root and a similar modification can be made.

(iii) If $\lambda_{112} = 0$, (4) shows that the images of elements of A_2 will contain no component in v_2, which can therefore be deleted from B_2 and the third row and column from $\hat{E}_2$. In this case $4\lambda_{011}^2$ is not a plenary train root.

(iv) However, if $\lambda_{022} = 2\lambda_{011}^2$, there is in general a genuine double plenary train root.

These details may be verified by calculation.

4. *Algebras of arbitrary rank*

In connection with A_n we consider that special train algebra A_{n-1} whose constants λ_{ijk} have the same values as those of A_n for $i, j, k = 0, 1, \ldots, n-1$.

THEOREM 1. *The correspondence H between A_n and A_{n-1} determined by*

$$(u_0, u_1, \ldots, u_{n-1}, u_n)\,H = (u_0, u_1, \ldots, u_{n-1})$$

is a homomorphism.

Proof. This may be verified on inspection, since in A_n, $c_i c_n$ contains no component in c_j, $j \leqslant n-1$,

THEOREM 2. *The plane of unit weight in A_n may be mapped into a variety V_n lying in a space B_n (generally not of the same dimension as A_n) by a correspondence R,*

$$(1, u_1, \ldots, u_n)\,R = (1, v_1, \ldots, v_m),$$

where $$v_i = u_{i_1}^{k_1} u_{i_2}^{k_2} \dots u_{i_n}^{k_n} \tag{6}$$

in such a way that a linear transformation of B_n with matrix $\hat{E}_n$ can be found having the following properties:

(i) *E_n operating on elements of unit weight in A corresponds to $\hat{E}_n$ operating on their images in B_n, in the sense that*

$$(xR)\,\hat{E}_n = (xE_n)\,R.$$

(ii) *$\hat{E}_n$ is upper triangular.*

(iii) *The proper value α_i of $\hat{E}_n$ such that $(xR)\,(\hat{E}_n - \alpha_i I)$ has no component in v_i given by* (6) *is*

$$(2\lambda_{0i_1 i_1})^{k_1}(2\lambda_{0i_2 i_2})^{k_2}\dots(2\lambda_{0i_n i_n})^{k_n}.$$

(iv) *For general values of the λ_{ijk}, the dimension of B_n is minimal among spaces fulfilling these conditions.*

Informally, $u_n E_n$ involves u_{n-1}^2. Hence if B_{n-1} has been found corresponding to A_{n-1}, and a mapping R given by (6), B_n will need to contain a dimension corresponding to each distinct product of powers given by multiplying pairs of expressions on the right of (6), and one corresponding to u_n. This involves a modified Kronecker square construction.

Proof. The results of §3 show that the thoerem is true for A_1 and A_2. Suppose that it is true for A_{n-1}. Let us denote the elements of B_{n-1} by $(v_0, v_1, \dots, v_{m'})$ and the elements of $\hat{E}_{n-1}$ by d_{ij}, $i, j = 0, \dots, m'$. The required space B_n and matrix $\hat{E}_n$ will be constructed in two stages, the intermediate constructions being called $\tilde{B}$ and $\tilde{E}$.

For $\tilde{B}$ we take a space of dimension $m + 1 = \frac{1}{2}(m'+1)\,(m'+2)+1$. The first m coordinates are formed from the Kronecker square of B_{n-1}, that is they are the products $v_i v_j$, $i = 0, \dots, m'$ and $j \geqslant i$, ordered so that $v_r v_s$ precedes $v_k v_l$ if either

$$r < k, \text{ or } r = k,\ s < l. \tag{7}$$

The $(m+1)$-th coordinate of $\tilde{B}$ is $v_m = u_n$. This definition implies the mapping $\tilde{R}$ of the unit plane of A_n into a variety in $\tilde{B}$. For the first m rows and columns of $\tilde{E}$ we take the Kronecker square of $\hat{E}_{n-1}$, which means that the element in the row corresponding to $v_r v_s$ is

$$d_{kr} d_{ls} + d_{ks} d_{lr}. \tag{8}$$

The $(m+1)$-th row and column of $\tilde{E}$ are defined by

$$d_{mj}=0, \quad j=0, \ldots, m-1, \tag{9a}$$

$$d_{mm}=2\lambda_{0nn}, \tag{9b}$$

$$\left.\begin{aligned} d_{km} &= 2\lambda_{ijn}, && \text{if the } k\text{-th row of } \tilde{E} \text{ expressed in terms of (6)} \\ & && \text{corresponds to } u_i u_j,\ i \neq j \\ &= \lambda_{iin}, && \text{if the } k\text{-th row of } \tilde{E} \text{ corresponds to } u_i^2 \\ &= 0 && \text{otherwise.} \end{aligned}\right\} \tag{9c}$$

$\tilde{E}$ satisfies (ii) and (iii) of Theorem 2. Since $\hat{E}_{n-1}$ is upper triangular by hypothesis, $d_{ij}=0$ if $i>j$. If $v_r v_s$ precedes $v_k v_l$ in the ordering given above, (7) and the inequality $1>r$ which it implies lead to the conclusions that (8) is zero for elements below the main diagonal of $\tilde{E}$, and that the element in the row and column corresponding to $v_k v_l$ is $d_{kk} d_{ll}$. Since the matrix is upper triangular this is the proper value corresponding to $v_k v_l$ and the induction hypothesis then leads to the verification of (iii). The addition of the last row and column as defined by (9) clearly does not affect the validity of (ii) and (iii).

In general, when the v_i are expressed in terms of the u_j by (6), it will happen that $v_k v_l = v_r v_s$ for some k, l, r, s, say with the first member coming first in the established ordering. By what has been proved, the proper values in the row and column corresponding to each of these coordinates will be equal. For each occurrence of this type, add the row of $\tilde{E}$ corresponding to $v_r v_s$ to that corresponding to $v_k v_l$, and delete the row and column corresponding to $v_r v_s$. Also delete the coordinate of $\tilde{B}$ corresponding to $v_r v_s$. Clearly, this procedure does not affect conditions (ii) and (iii), its only effect on the set of proper values being to eliminate multiplicities. Further, since the last column of $\tilde{E}$ has non-zero entries only for coordinates corresponding to $u_i u_j$, the elements of this column are not affected by the reduction procedure exept for relabeling of their row numbers. The results of the reduction procedure are the required space B_n and matrix $\hat{E}_n$.

To prove that (i) is true for A_n, B_n, E_n and $\hat{E}_n$ consider the square of an element of unit weight $x \in A_n$, and let xH be its image in the homomorphic mapping of A_n into the A_{n-1} corresponding to it, of Theorem 1.

$$\begin{aligned} xE_n &= (xH + u_n c_n)^2 \\ &= (xH)\, E_{n-1} + \left\{ 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} \lambda_{ijn} u_i u_j + \sum_{i=1}^{n-1} \lambda_{iin} u_i^2 + 2\lambda_{0nn} u_n \right\} c_n. \end{aligned} \tag{10}$$

The first term, where E_{n-1} is an operator with domain and range A_{n-1}, shows that the appropriate transform of the first m coordinates of x is the Kronecker square of $\hat{E}_{n-1}$, reduced to allow for identities among the $v_i v_j$. The remainder of (10) shows that (9b, c) gives the required last column of $\hat{E}_n$.

To prove that it is not in general possible to find a space of fewer dimensions in which a linear operator can be made to correspond to E_n, consider the square of an element of unit weight as given by (10):

$$u_n E_n = \lambda_{n-1,\, n-1,\, n}\, u_{n-1}^2 + \dots .$$

By hypothesis, $u_{n-1} E_{n-1}$ generates all the distinct products of powers of the u_i involved in the right-hand sides of (6). Hence

$$u_{n-1}^2 E_{n-1} = (u_{n-1} E_{n-1})^2$$

will generate those corresponding to distinct values of the $v_i v_j$. Hence after the reductions of the above paragraph, no further reduction of B_n is possible. This completes the proof.

For particular sets of the λ_{ijk}, spaces of smaller dimension than the B_n just constructed may satisfy the conditions of the theorem. An example is given in (iii) at the end of §3.

THEOREM 3. *Plenary powers in A_n form a train. The plenary train roots of A_n are included in the following set: the products taken in pairs of those of the A_{n-1} to which A_n corresponds in the homomorphism of Theorem* 1, *including squares, and* $2\lambda_{0nn}$.

Proof. This follows from the relation between the proper values of a Kronecker square and those of its exponand, [**8**; vol. I, p. 75] together with (9a, b) and the fact that the reduction procedure of Theorem 2 only removes superfluous multiplicities among the proper values.

5. *Relations with previous work*

Plenary train equations for algebras associated with specific genetic situations have been derived by Etherington [**3**, **5**], by the method of annulling polynomials. The method used here involves a formalisation of this. The idea of linearising the quadratic recurrence relationships arising in population genetics, between gametic frequencies at successive generations, by introducing polynomials in the frequencies which themselves satisfy linear equations, was used by Haldane [**10**] for polyploidy, and Bennett [**1**] for linked loci. Bennett's principal components correspond to the proper vectors of $\hat{E}_n$ here. In surveying their work, Moran [**12**; p. 38] asked under what conditions such a procedure was possible, and Theorem 3 provides a sufficient condition, namely that the genetic situation should correspond to a special train algebra. The restriction to quadratic functions which Moran mentions does not however seem possible for algebras of rank > 4, and the impression that Haldane only required quadratics arises from the mistake in his algebra, which Moran has corrected [**12**; p. 41]. Reiersøl [**13**], like Bennett, dealt with the case of n linked loci in a recursive way, making use of the genetic symmetry rather than transforming to a canonical basis. Reiersøl's method makes use of simultaneous homomorphisms onto algebras of considerably smaller dimension, where my method uses a single homorphism onto an algebra of dimension one fewer than that being studied.

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